Perfect Subsets of the Unit Interval: Some Surprising Results about the Real Numbers

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Chipsets

What are chipsets?

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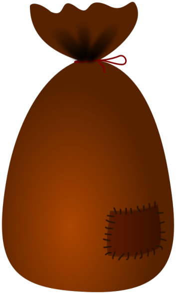

Chipsets

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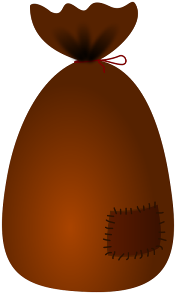

https://1001freedownloads.s3.amazonaws.com/vector/thumb/77370/bag.png
Chips: https://pixy.org/979183/

Chipsets

One chip for each positive integer:

...

https://teamgolfusa.com/wp-content/uploads/2016/03/red-large.png

Chipsets

Think of chips as positive integers, and bags of chips as sets containing positive integers.

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Examples:

$\{1, 2, 3, 4\}$ is a bag with chips 1-4 in it.

$\mathbb{Z}^+$ is a bag containing every chip.

However, these are not chipsets.

Chipsets

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- Step $n = 2$: Add 4 chips and remove 2 chips.
- Step $n = 3$: Add 8 chips and remove 4 chips.
- ...
- Step n: Add 2^n chips and remove 2^{n-1} chips.
- ...

Note: when removing chips, you can remove *any* of the chips that are currently in the bag.

Chipsets

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  then remove 4 chips $\rightarrow \{3, 6, 7, 8, 11, 13, 14\}$
- ...

Informally, what we get "in the end" is a chipset.

Chipsets

Some examples:

- The empty set

Chipsets

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- Any singleton set like $\{5\}$ or $\{2019\}$

Chipsets

Some examples:

- The empty set
- Any singleton set like $\{5\}$ or $\{2019\}$
- The set $\{1, 3, 5, 7, 9, 11, 13\}$
- More interesting examples soon...

Chipsets

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- The set $\{3, 4, 5, 6\}$ let's see why:
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- The set $\{3, 4, 5, 6\}$ let's see why:
  - Step 1: $\{1, 2\} \rightarrow \{1\}$
  - Step 2: $\{1, 3, 4, 5, 6\} \rightarrow \{4, 5, 6\}$

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- The set $\{3, 4, 5, 6\}$ let's see why:
  - Step 1: $\{1, 2\} \to \{1\}$
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- $\mathbb{Z}^+$

Chipsets

Let's develop a formal definition for a chipset. First, we make the following definition:

Definition: The chip counting function

Define $\kappa(n)$ to be the total number of chips added to a bag after step n. For $n \in \mathbb{Z}^+$, a geometric series calculation shows $\kappa(n) = 2^{n+1} - 2$.

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To see this in action:

- $\kappa(1) = 2$
- $\kappa(2) = 6$
- $\kappa(3) = 14$

These numbers will be important to us.

Chipsets

We'll also need to introduce a bit of notation:

Notation

Let $X \subseteq \mathbb{Z}^+$. Given a positive integer n, denote $X[n] = X \cap \{1, ..., n\}$.

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Some examples to illustrate this:

- $\{1, 2, 3, 4\}[2] = \{1, 2\}$
- Let E be the set of even positive integers. Then $E[7] = \{2, 4, 6\}$.

Chipsets

We can now formally define a chipset:

Definition: Chipset

Let $X \subseteq \mathbb{Z}^+$. We say X is a *chipset* provided $\left|X[\kappa(n)]\right| \leq \frac{\kappa(n)}{2}$ for all positive integers n.

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Let's unpack this with some examples:

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  - $n = 1$: $\left|\{1,3,4\}[\kappa(1)]\right| = \left|\{1,3,4\}[2]\right| = \left|\{1\}\right| = 1 \leq \frac{2}{2} = \frac{\kappa(1)}{2}$

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- The positive evens and positive odds are chipsets by this definition.

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- The positive evens and positive odds are chipsets by this definition.
- $\{1,2\}$ fails because $\left|\{1,2\}[\kappa(1)]\right| = \left|\{1,2\}[2]\right| = 2 > \frac{2}{2} = \frac{\kappa(1)}{2}$.

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It is also useful to separate binary representations with spaces to indicate which chips were added at which steps. For example, the chipset $\{1, 3, 4, 7, 10\}$ could be represented by 10 1100 10010000.

$$\underbrace{10}_{\text{n=1}}\ \underbrace{1100}_{\text{n=2}}\ \underbrace{10010000}_{\text{n=3}}$$

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Now all we have to do to verify that a set is a chipset is count the number of 1s in its binary representation. For example, we could determine whether 10 0010 10100100 is a chipset as follows:

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This means 10 0010 10100100 does indeed represent a chipset, in fact, it represents $\{1, 5, 7, 9, 12\}$.

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$$\chi(k) = \begin{cases} 1 & k \in X \\ 0 & k \notin X \end{cases}$$

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This function will tell us if a chip is in a bag. For example, if $X = \{1, 3, 4\}$ then $\chi(1) = 1$, $\chi(2) = 0$, $\chi(3) = 1$, $\chi(4) = 1$, and $\chi(n) = 0$ for all $n > 4$.

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Let's now define how to "weigh" sets of integers:

Definition: Weighting function

Let $X \subseteq \mathbb{Z}^+$. Define

$$\rho(X) = \sum_{n=1}^{\infty} \frac{\chi(n)}{2^n}$$

where χ is the characteristic function of X.

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- $\rho(\mathbb{Z}^+) = 1$ (this is the largest possible output)
  - $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

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- $\rho(\{1, 2\}) = \frac{1}{2} + \frac{1}{4} + \frac{0}{8} + \frac{0}{16} + \ldots = \frac{3}{4}$
- $\rho(\varnothing) = 0$ (this is the smallest possible output)

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We finally define chip numbers:

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If X is a chipset, $\rho(X)$ is a chip number.

Examples:

- $\{1, 2\}$ is not a chipset, so $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ is *not* a chip number.
- The set of primes is a chipset, and its chip number is approximately 0.414683.

Chip Numbers

Given a binary representation of a chipset, computing its chip number is straightforward.

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Given a binary representation of a chipset, computing its chip number is straightforward. Example:

$$01\ 0101\ 00100100$$

has as its chip number

$$\frac{0}{2}+\frac{1}{2^2}+\frac{0}{2^3}+\frac{1}{2^4}+\frac{0}{2^5}+\frac{1}{2^6}+\frac{0}{2^7}+\frac{0}{2^8}+\frac{1}{2^9}+\frac{0}{2^{10}}+\frac{0}{2^{11}}+\frac{1}{2^{12}}+\frac{0}{2^{13}}+\frac{0}{2^{14}}$$

$$\approx 0.33032$$

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$$\approx 0.33032$$

Interesting fact: in base 2, this is equivalent to simply inserting a decimal at the beginning of the representation. Thus, our chip number can be written as

$$.01010100100100$$

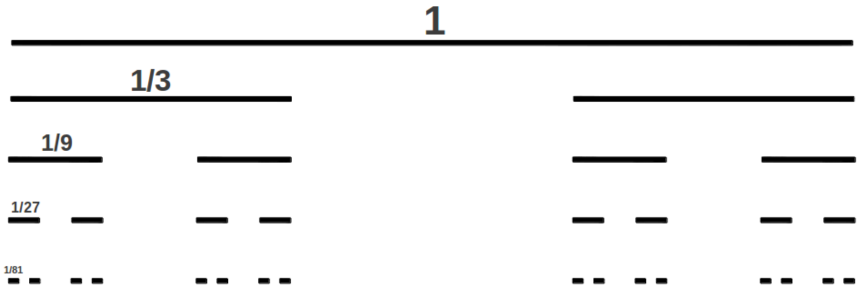
in binary.

Activity Time

Let's make some chipsets and compute some chip numbers.

Cantor Sets

Why do we care about all this? First, let's look at a pretty cool set:



Cantor Middle-Thirds Set

From: `https://blogs.sas.com/content/iml/2016/06/29/viz-cantor-function-in-sas.html`

Cantor Sets

We can compute how much was “removed” using a geometric series:

$$\frac{1}{3}$$

Cantor Sets

We can compute how much was "removed" using a geometric series:

$$\frac{1}{3} + \frac{2}{9}$$

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We can compute how much was “removed” using a geometric series:

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Cantor Sets

We can compute how much was “removed” using a geometric series:

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + ... = \frac{1}{3}\left(\frac{1}{1 - \frac{2}{3}}\right) = 1$$

So everything got removed! Or did it? What about the endpoints?

Cantor Sets

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So everything got removed! Or did it? What about the endpoints? This set has *measure zero*.

Fat Cantor Sets

It gets even more weird. Let's look at a similar construction:

Fat Cantor Set

From: https://en.wikipedia.org/wiki/Smith-Volterra-Cantor_set

Fat Cantor Sets

We can again compute how much was "removed" using a geometric series:

$$\frac{1}{4}$$

Fat Cantor Sets

We can again compute how much was “removed” using a geometric series:

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Fat Cantor Sets

We can again compute how much was “removed” using a geometric series:

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So there are no intervals remaining, but only half of the unit-interval was removed.

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So there are no intervals remaining, but only half of the unit-interval was removed. This set has *measure one-half*. It "takes up space" yet has no intervals.

Cantor Sets

Cantor sets have the following properties in common:

- Non-empty
- Compact
  - In the reals this means closed and bounded.
- Perfect
  - Closed and every point is a limit-point.
- Totally disconnected
  - The only connected subsets are the empty set and the one-point sets. A familiar example is the set of rational numbers.
- Metrizable
  - This means there is a notion of "distance" between points. This is always true in the reals.

The Set of all Chip Numbers

Let's look at a very weird set, the set of all chip numbers:

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Define $\mathcal{C} = \{x \in [0, 1] \mid x = \rho(X)$ for some chipset X$\}$.

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What properties does $\mathcal{C}$ have?

- Non-empty, because $\frac{1}{2}$ is a chip number.

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Definition: The Set of all Chip Numbers

Define $\mathcal{C} = \{x \in [0,1] \mid x = \rho(X) \text{ for some chipset X}\}$.

What properties does $\mathcal{C}$ have?

- Non-empty, because $\frac{1}{2}$ is a chip number.
- Bounded, because $\mathcal{C}$ lives in the unit-interval.

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- Perfect, because every chip number has a sequence of chip numbers (not containing it) that converges to it.
- Totally disconnected, because $\mathcal{C}$ contains no intervals, since given any two chip numbers, there exists a number between them which is not a chip number.

The Set of all Chip Numbers

Since $\mathcal{C}$ satisfies the properties in the previous slide...

The Set of all Chip Numbers

Since $\mathcal{C}$ satisfies the properties in the previous slide... $\mathcal{C}$ is a Cantor space by Brouwer's Theorem. This means there is a homeomorphism between $\mathcal{C}$ and the Cantor set.

References

Images:

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Paper from Dr. Aaron Montgomery