Computing Free Resolutions of OI-Modules

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Outline

- Brief introduction to free resolutions
- Review of Gröbner basis theory for OI-modules
- Free resolutions of OI-modules
- The OI-Schreyer's Theorem

Some motivation

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- Let M be a module over a commutative Noetherian ring R
- Let $m_1, \ldots, m_n \in M$
- Do the m_i form an R-linearly independent set?
- If not, how "far" are the m_i from being R-linearly independent?

Syzygies

Consider the map $\varphi : \bigoplus_{i=1}^{n} Re_i \to \langle m_1, \ldots, m_n \rangle$ given by $e_i \mapsto m_i$. The kernel of φ is called the *(first) syzygy module* of $m_1, \ldots, m_n$. Elements of $\ker(\varphi)$ are called *syzygies* and correspond to R-linear relations on the m_i.

Remark

We have $\ker(\varphi) = 0$ if and only if the m_i are R-linearly independent.

Our running example:

$$\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z} \to 0$$

where φ is given by $1 \mapsto 1 + 2\mathbb{Z}$. One checks that $\ker(\varphi) = 2\mathbb{Z}$.

But wait, there's more...

Since R is Noetherian, $\bigoplus_{i=1}^{n} Re_i$ is a Noetherian module. So $\ker(\varphi)$ is finitely generated, say by $s_1, \ldots, s_m$. Do the s_i form an R-linearly independent set?

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We repeat the process: take the map $\psi : \bigoplus_{i=1}^{m} Rd_i \to \langle s_1, \ldots, s_m \rangle$ given by $d_i \mapsto s_i$. Then $\ker(\psi)$ is called the *(second) syzygy module* of $m_1, \ldots, m_n$.

Note: by construction, we have $\text{im}(\psi) = \ker(\varphi)$.

Our running example:

$$\mathbb{Z} \xrightarrow{\psi} \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z} \to 0$$

where ψ is given by $1 \mapsto 2$. One checks that $\ker(\psi) = 0$.

Free resolutions

Continuing in this way, we obtain an exact sequence

$$\cdots \to \bigoplus_{i=1}^{m} Rd_i \xrightarrow{\psi} \bigoplus_{i=1}^{n} Re_i \xrightarrow{\varphi} \langle m_1, \ldots, m_n \rangle \to 0.$$

Free resolutions

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This is an example of a *free resolution* of a module M, i.e. an exact sequence of the form

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

where each F_i is a free R-module.

Our running example is a *finite* free resolution:

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z} \to 0$$

What can you do with free resolutions?

To list a few things...

- Homological constructions such as Ext and Tor
- Hilbert functions and Hilbert polynomials
- Betti numbers

A theorem of Hilbert

Theorem (Hilbert's Syzygy Theorem)

Every finitely generated module over the polynomial ring $K[x_1, \ldots, x_n]$ has a finite free resolution with length $\leq n$.

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A proof due to Schreyer (see [E95, Chapter 15]) gives explicit generators for the syzygy modules. This lets one compute free resolutions in *finite time*.

What about sequences of modules?

Given a sequence of related modules over a sequence of related polynomial rings, we wish to simultaneously compute a free resolution for each module.

To formalize the notion of "related sequence" we use the framework of OI-modules over OI-algebras.

OI-algebras

- Let $[n] = \{1, \ldots, n\}$ for $n \in \mathbb{N}$ and set $[0] = \emptyset$.
- Let OI denote the category whose objects are $[n]$ for $n \in \mathbb{Z}_{\geq 0}$ and whose morphisms are strictly increasing maps.
- Let K be a field, and denote by K-Alg the category of commutative, associative, unital K-algebras whose morphisms are unital K-algebra homomorphisms.

Definition

An OI-*algebra* over K is a covariant functor $\mathbf{A} : \mathrm{OI} \to K\text{-Alg}$.

- For an object $[n] \in \mathrm{OI}$ and any functor F out of OI, we write F_n instead of $F([n])$. We call F_n the *width n component* of F.
- We write $\mathrm{Hom}(m, n)$ instead of $\mathrm{Hom}_{\mathrm{OI}}([m], [n])$.

Our main OI-algebra

Example

Fix $c \in \mathbb{N}$ and define an OI-algebra $\mathbf{P}$ as follows:

1. For $m \geq 0$ set
$$\mathbf{P}_m = K[x_{i,j} \ : \ i \in [c],\ j \in [m]].$$
2. For $\varepsilon \in \mathrm{Hom}(m, n)$ define
$$\mathbf{P}(\varepsilon) : \mathbf{P}_m \to \mathbf{P}_n \quad \text{via} \quad x_{i,j} \mapsto x_{i,\varepsilon(j)}.$$

Example

If $c = 2$, we can think of $\mathbf{P}$ as the sequence

$$K, K\begin{bmatrix} x_{1,1} \\ x_{2,1} \end{bmatrix}, K\begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix}, K\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \end{bmatrix}, \ldots$$

Continuing the example...

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Now let $\varepsilon \in \mathrm{Hom}(2, 3)$ be given by $1 \mapsto 2$ and $2 \mapsto 3$.

Let $f = 3x_{2,1}^2 - x_{1,1}x_{2,2} \in \mathbf{P}_2$.

Then $\mathbf{P}(\varepsilon)(f) = 3x_{2,2}^2 - x_{1,2}x_{2,3} \in \mathbf{P}_3$.

OI-modules

Definition

An OI-*module* over an OI-algebra $\mathbf{A}$ is a covariant functor $\mathbf{M} : \mathrm{OI} \to K\text{-Vect}$ such that

1. each $\mathbf{M}_m$ is an $\mathbf{A}_m$-module, and
2. for each $\varepsilon \in \mathrm{Hom}(m, n)$ and any $a \in \mathbf{A}_m$, the diagram

$$\begin{array}{ccc} \mathbf{M}_m & \xrightarrow{\mathbf{M}(\varepsilon)} & \mathbf{M}_n \\ {\scriptstyle a\cdot}\downarrow & & \downarrow{\scriptstyle \mathbf{A}(\varepsilon)(a)\cdot} \\ \mathbf{M}_m & \xrightarrow{\mathbf{M}(\varepsilon)} & \mathbf{M}_n \end{array}$$

   commutes.

We often refer to $\mathbf{M}$ as an $\mathbf{A}$-*module*.

OI-submodules

Definition

A *subset* of an OI-module $\mathbf{M}$ is a subset of $\coprod_{m\geq 0}\mathbf{M}_m$. An *element* of $\mathbf{M}$ is an element of $\mathbf{M}_m$ for some $m \geq 0$. Such an element has *width* m. If $\mathbf{M}$ and $\mathbf{N}$ are OI-modules, then by $\mathbf{N} \subseteq \mathbf{M}$ we mean $\mathbf{N}_m \subseteq \mathbf{M}_m$ for all $m \geq 0$.

Definition

Let $\mathbf{M}$ and $\mathbf{N}$ be OI-modules. We say $\mathbf{N}$ is a *submodule* of $\mathbf{M}$ if $\mathbf{N} \subseteq \mathbf{M}$ and $\mathbf{N}$ inherits its structure from $\mathbf{M}$.

Orbits and generation

Let $G \subseteq \mathbf{M}$ and let $m \geq 0$. The *m-orbit* of G is the set

$$\mathrm{Orb}(G, m) = \{\mathbf{M}(\varepsilon)(g) \ : \ g \in \mathbf{M}_\ell \cap G, \ \varepsilon \in \mathrm{Hom}(\ell, m)\}.$$

Definition

A submodule $\mathbf{N} \subseteq \mathbf{M}$ is *finitely generated* if there is a finite subset $G \subset \mathbf{N}$ such that $\mathbf{N}_m = \langle \mathrm{Orb}(G, m) \rangle$ for all $m \geq 0$. In this case we write $\mathbf{N} = \langle G \rangle_{\mathbf{M}}$.

Example

If we consider $\mathbf{P}$ as an OI-module over itself, then the OI-*ideal* given by $\mathbf{I}_n = \langle x_i x_j \ : \ 1 \leq i < j \leq n \rangle$ is finitely generated by $\{x_1 x_2\}$.

Homomorphisms

Definition

Let $\mathbf{M}$ and $\mathbf{N}$ be $\mathbf{A}$-modules. A *homomorphism* (or $\mathbf{A}$-linear map) $\varphi : \mathbf{M} \to \mathbf{N}$ is a collection of $\mathbf{A}_n$-linear maps $\varphi_n : \mathbf{M}_n \to \mathbf{N}_n$ such that the diagram

$$\begin{array}{ccc} \mathbf{M}_m & \xrightarrow{\varphi_m} & \mathbf{N}_m \\ {\scriptstyle \mathbf{M}(\varepsilon)}\downarrow & & \downarrow{\scriptstyle \mathbf{N}(\varepsilon)} \\ \mathbf{M}_n & \xrightarrow{\varphi_n} & \mathbf{N}_n \end{array}$$

commutes for all $\varepsilon \in \mathrm{Hom}(m, n)$. In categorical terms, φ is a natural transformation such that each φ_n is $\mathbf{A}_n$-linear.

Example

Let $\varphi : \mathbf{M} \to \mathbf{N}$ be an $\mathbf{A}$-linear map. Then $\ker(\varphi)$ is the submodule of $\mathbf{M}$ given by $(\ker(\varphi))_n = \ker(\varphi_n)$. Similarly, $\mathrm{im}(\varphi)$ is the submodule of $\mathbf{N}$ given by $(\mathrm{im}(\varphi))_n = \mathrm{im}(\varphi_n)$.

Free OI-modules

Definition

Fix integers $d_1, \ldots, d_s \geq 0$ and define an OI-module over $\mathbf{P}$ as follows. For all $n \geq 0$ let

$$\mathbf{F}_n = \bigoplus_{\substack{1 \leq i \leq s \\ \pi \in \mathrm{Hom}(d_i, n)}} \mathbf{P}_n e_{\pi,i}$$

and for all $\varepsilon \in \mathrm{Hom}(m, n)$ define $\mathbf{F}(\varepsilon) : \mathbf{F}_m \to \mathbf{F}_n$ via $e_{\pi,i} \mapsto e_{\varepsilon \circ \pi, i}$. We call $\mathbf{F}$ a *free* OI-*module with basis* $\{e_{\mathrm{id}_{[d_i]}, i}\}$.

Remark

Any $\mathbf{P}$-linear map out of $\mathbf{F}$ is uniquely determined by where the $e_{\mathrm{id}_{[d_i]}, i}$ are sent.

Free OI-module example

Example

Let $\mathbf{F}$ have basis $\{e_{\mathrm{id}_{[2]}}\}$. Then $\mathbf{F}_n$ is a free $\mathbf{P}_n$-module of rank $\binom{n}{2}$ for all $n \geq 0$.

Specifically, if $c = 1$ we can think of $\mathbf{F}$ as the sequence

$$0, 0, K[x_1, x_2], K[x_1, x_2, x_3]^3, K[x_1, x_2, x_3, x_4]^6, \ldots$$

Monomial orders

A monomial in $\mathbf{F}$ is an element $x^u e_{\pi,i}$ for some monomial x^u in $\mathbf{P}$.

Definition

A total order $<$ on the monomials of $\mathbf{F}$ is a *monomial order* on $\mathbf{F}$ if for all monomials $\mu, \nu \in \mathbf{F}_m$ with $\mu < \nu$ we have

1. $\mu < a\mu < a\nu$ for all monomials $1 \neq a \in \mathbf{P}_m$, and
2. $\mu < \mathbf{F}(\varepsilon)(\mu) < \mathbf{F}(\varepsilon)(\nu)$ for all $\varepsilon \in \mathrm{Hom}(m, n)$ with $m < n$.

Monomial orders exist, for example the *lex order*, and any monomial order is a *well-order* [NR19].

Division with remainder

Given a monomial order $<$ on $\mathbf{F}$ and an element $f \in \mathbf{F}_m$, one defines $\mathrm{lm}(f) \in \mathbf{F}_m$, $\mathrm{lt}(f) \in \mathbf{F}_m$ and $\mathrm{lc}(f) \in K$.

Remark

Any monomial order $<$ on $\mathbf{F}$ restricts to a monomial order $<_n$ on $\mathbf{F}_n$ for all $n \geq 0$.

Definition

Let $f \in \mathbf{F}_n$ and let $G \subseteq \mathbf{F}$. A *remainder of f modulo G* (with respect to $<$) is defined to be a remainder of f modulo $\mathrm{Orb}(G, n)$ (with respect to $<_n$).

Remainders of f modulo G can be computed in finite time [CLO].

Division with remainder

What does it mean to be a remainder?

If r is a remainder of $f \neq 0$ modulo G, then we can write $f = \sum a_i q_i + r$ for some $a_i \in \mathbf{P}_n$ and some $q_i \in \mathrm{Orb}(G, n)$ such that

- either $r = 0$ or $\mathrm{lm}(r)$ is not divisible by any element of $\mathrm{Orb}(\mathrm{lm}(G), n)$,
- $\mathrm{lm}(r) < \mathrm{lm}(f)$ if $r \neq 0$, and
- $\mathrm{lm}(a_i q_i) \leq \mathrm{lm}(f)$ whenever $a_i q_i \neq 0$.

Gröbner bases

Definition

Fix a monomial order $<$ on $\mathbf{F}$ and let $\mathbf{M}$ be a submodule of $\mathbf{F}$. A subset $G \subseteq \mathbf{M}$ is called a *Gröbner basis* for $\mathbf{M}$ if

$$\langle \operatorname{lm}(\mathbf{M}) \rangle_{\mathbf{F}} = \langle \operatorname{lm}(G) \rangle_{\mathbf{F}}$$

where $\operatorname{lm}(\mathbf{M}) = \{\operatorname{lm}(f) \ : \ f \in \mathbf{M}\}$ and $\operatorname{lm}(G) = \{\operatorname{lm}(g) \ : \ g \in G\}$.

Remark

A set $G \subseteq \mathbf{M}$ forms a Gröbner basis for $\mathbf{M}$ if and only if $\operatorname{Orb}(G, n)$ forms a Gröbner basis for $\mathbf{M}_n$ with respect to $<_n$ for all $n \geq 0$.

S-Polynomials and Critical Pairs

Definition

The *S-polynomial* of $f, g \in \mathbf{F}_m$ is the combination

$$S(f,g) = \frac{\operatorname{lcm}(\operatorname{lm}(f), \operatorname{lm}(g))}{\operatorname{lt}(f)} f - \frac{\operatorname{lcm}(\operatorname{lm}(f), \operatorname{lm}(g))}{\operatorname{lt}(g)} g.$$

Definition

Let $B \subseteq \mathbf{F}$ and let $m \geq 0$. A tuple

$$(\mathbf{F}(\sigma)(f), \mathbf{F}(\tau)(g)) \in \operatorname{Orb}(B,m) \times \operatorname{Orb}(B,m)$$

is called a *critical pair* if $\operatorname{lm}(f)$ and $\operatorname{lm}(g)$ involve the same basis element and $m = |\operatorname{im}(\sigma) \cup \operatorname{im}(\tau)|$. The set of all critical pairs of B is denoted $\mathcal{C}(B)$.

Important: $\mathcal{C}(B)$ is finite if B is finite.

OI-Buchberger's Criterion

Theorem (M, Nagel)

A generating set G of a submodule $\mathbf{M}$ *of* $\mathbf{F}$ *forms a Gröbner basis for* $\mathbf{M}$ *if and only if each* $S(f, g)$ *with* $(f, g) \in \mathcal{C}(G)$ *has a remainder of zero modulo* G.

Key idea: any S-polynomial $S(\mathbf{F}(\sigma)(f), \mathbf{F}(\tau)(g))$ can be written as $\mathbf{F}(\rho)(S(\mathbf{F}(\overline{\sigma})(f), \mathbf{F}(\overline{\tau})(g)))$ where $(\mathbf{F}(\overline{\sigma})(f), \mathbf{F}(\overline{\tau})(g))$ is a critical pair.

Buchberger's Algorithm

Let $<$ be a monomial order on $\mathbf{F}$ and let $G \subset \mathbf{F}$ be a finite set.

1. Are there $(f, g) \in \mathcal{C}(G)$ such that $S(f, g)$ has a nonzero remainder modulo G?
2. If so, append the remainder to G and repeat.
3. $\mathbf{F}$ is Noetherian (see [NR19]) so this process terminates.
4. Computes a finite Gröbner basis for $\langle G \rangle_{\mathbf{F}}$.

Gröbner basis example

Consider $\mathbf{P}$ as a free OI-module over itself. Let $c = 2$ and let $B = \{x_{2,1}^2 + x_{1,1} \in \mathbf{P}_1, x_{2,2} + x_{1,2}x_{1,1} \in \mathbf{P}_2\}$. Using Macaulay2, we can compute a Gröbner basis for $\langle B \rangle_{\mathbf{P}}$. It consists of the elements

$$\begin{aligned} x_{2,1}^2 + x_{1,1} &\in \mathbf{P}_1 \\ x_{2,2} + x_{1,2}x_{1,1} &\in \mathbf{P}_2 \\ x_{1,2}^2x_{1,1}^2 + x_{1,2} &\in \mathbf{P}_2 \\ -x_{1,3}^2 + x_{1,3}x_{1,2} &\in \mathbf{P}_3 \\ -x_{1,3}x_{1,2} + x_{1,3}x_{1,1} &\in \mathbf{P}_3 \\ -x_{1,3}x_{1,1}^3 - x_{1,3} &\in \mathbf{P}_3 \end{aligned}$$

https://github.com/morrowmh/OIGroebnerBases

Return to free resolutions

Definition

Let $\mathbf{M}$ be an $\mathbf{A}$-module. A *free resolution* of $\mathbf{M}$ is an exact sequence

$$\cdots \to \mathbf{F}^2 \to \mathbf{F}^1 \to \mathbf{F}^0 \to \mathbf{M} \to 0$$

where each $\mathbf{F}^i$ is a free OI-module over $\mathbf{A}$.

Theorem (Nagel, Römer, 2019)

If $\mathbf{M}$ is a finitely generated $\mathbf{P}$-module, then a free resolution of $\mathbf{M}$ exists where each $\mathbf{F}^i$ is finitely generated.

Restricting to a width

If we can find a resolution

$$\cdots \to \mathbf{F}^2 \to \mathbf{F}^1 \to \mathbf{F}^0 \to \mathbf{M} \to 0$$

then for all $w \geq 0$ we get an induced free resolution

$$\cdots \to \mathbf{F}^2_w \to \mathbf{F}^1_w \to \mathbf{F}^0_w \to \mathbf{M}_w \to 0$$

over $\mathbf{P}_w$.

A word on minimal resolutions

Remark

If $\mathbf{M}$ is graded, one can find a *graded* free resolution of $\mathbf{M}$, i.e. each map is degree preserving.

Theorem (Fieldsteel, Nagel, 2021)

Let $\mathbf{M}$ be a finitely generated graded $\mathbf{P}$-module. Then a minimal graded free resolution of $\mathbf{M}$ exists and is unique up to isomorphism.

Remark

Width-wise minimal implies minimal, but the converse does not hold in general.

From Gröbner bases to syzygies

Let **F** be a free OI-module over **P** and let **M** be a finitely generated submodule. We wish to compute a free resolution of **M**. Here is the process:

1. Use the OI-Buchberger's Algorithm to compute a finite Gröbner basis $B = \{b_1, \ldots, b_t\}$ of **M**.
2. Assume each $b_i \in \mathbf{M}_{w_i}$ and let **G** be the free **P**-module with basis $\{\epsilon_{\mathrm{id}_{[w_i]}, i}\}$.
3. Consider the **P**-linear map $\varphi : \mathbf{G} \to \langle B \rangle_{\mathbf{F}}$ induced by $\epsilon_{\mathrm{id}_{[w_i]}, i} \mapsto b_i$.
4. Use the OI-Schreyer's Theorem to compute a finite Gröbner basis for $\mathrm{Syz}(B) := \ker(\varphi)$.
5. Repeat.

Schreyer monomial order

Definition

1. Define a total order $\prec_B$ on the set

$$\{(\pi, i) \ : \ i \in [t], \ \pi \in \mathrm{Hom}(w_i, m), \ m \geq w_i\}$$

as follows. For $\pi \in \mathrm{Hom}(w_i, m)$ and $\rho \in \mathrm{Hom}(w_i, n)$ we say $\pi < \rho$ if

$$(m, \pi(1), \dots, \pi(w_i)) < (n, \rho(1), \dots, \rho(w_i))$$

in the usual lex order on $\mathbb{N}^{w_i+1}$. Now define $(\pi, i) \prec_B (\rho, j)$ if either $i < j$ or $i = j$ and $\pi < \rho$.

2. Define a total order $<_B$ on the monomials of $\mathbf{G}$ by setting $a\epsilon_{\pi,i} <_B b\epsilon_{\rho,j}$ if either $\mathrm{lm}(\varphi(a\epsilon_{\pi,i})) < \mathrm{lm}(\varphi(b\epsilon_{\rho,j}))$ or equality occurs and $(\rho, j) \prec_B (\pi, i)$.

Some setup...

Definition

For any $i, j \in [t]$, $\sigma \in \text{Hom}(w_i, m)$ and $\tau \in \text{Hom}(w_j, m)$ with $m \geq \max(w_i, w_j)$, use the division algorithm to write

$$S(\mathbf{F}(\sigma)(b_i), \mathbf{F}(\tau)(b_j)) = \sum_{\ell} a_{i,j,\ell}^{\sigma,\tau} \mathbf{F}(\pi_{i,j,\ell}^{\sigma,\tau})(b_{k_{i,j,\ell}^{\sigma,\tau}})$$

for some $a_{i,j,\ell}^{\sigma,\tau} \in \mathbf{P}_m$ and $\mathbf{F}(\pi_{i,j,\ell}^{\sigma,\tau})(b_{k_{i,j,\ell}^{\sigma,\tau}}) \in \text{Orb}(B, m)$. Define

$$s_{i,j}^{\sigma,\tau} = m_{i,j}^{\sigma,\tau} \epsilon_{\sigma,i} - m_{j,i}^{\tau,\sigma} \epsilon_{\tau,j} - \sum_{\ell} a_{i,j,\ell}^{\sigma,\tau} \epsilon_{\pi_{i,j,\ell}^{\sigma,\tau}, k_{i,j,\ell}^{\sigma,\tau}} \in \mathbf{G}$$

where

$$m_{i,j}^{\sigma,\tau} = \frac{\text{lcm}(\mathbf{F}(\sigma)(\text{lm}(b_i)), \mathbf{F}(\tau)(\text{lm}(b_j)))}{\mathbf{F}(\sigma)(\text{lm}(b_i))} \in \mathbf{P}_m.$$

OI-Schreyer's Theorem

Theorem (M,Nagel)

The $s_{i,j}^{\sigma,\tau}$ *with* $(\mathbf{F}(\sigma)(b_i), \mathbf{F}(\tau)(b_j)) \in \mathcal{C}(B)$ *and* $(\sigma, i) \prec_B (\tau, j)$ *form a finite Gröbner basis for* $\operatorname{Syz}(B)$ *with respect to* $<_B$*.*

Syzygy example

Let $\mathbf{G}$ be the free $\mathbf{P}$-module $(c = 2)$ with basis $\{\epsilon_{\mathrm{id}_{[2]}}\}$ and let $B = \{x_{1,1}x_{2,2} - x_{1,2}x_{2,1} \in \mathbf{P}_2\}$ so that $(\langle B \rangle_{\mathbf{P}})_n$ is the ideal of $\mathbf{P}_n$ generated by the 2×2 minors of the matrix

$$\begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \end{bmatrix}.$$

Using Macaulay2 we compute a Gröbner basis for $\mathrm{Syz}(B)$:

$$-x_{1,1}\epsilon_{23} + x_{1,2}\epsilon_{13} - x_{1,3}\epsilon_{12} \in \mathbf{G}_3$$

$$x_{2,2}\epsilon_{13} - x_{2,3}\epsilon_{12} - x_{2,1}\epsilon_{23} \in \mathbf{G}_3.$$

Note: $\epsilon_{ij} \leftrightsquigarrow \epsilon_\pi$ where $\pi : [2] \to [3]$ is given by $1 \mapsto i$ and $2 \mapsto j$.

`https://github.com/morrowmh/OIGroebnerBases`

Resolution example

Let $c = 1$ and let $\mathbf{F}$ be the free $\mathbf{P}$-module with basis $\{e_{\mathrm{id}_{[1]}}, e_{\mathrm{id}_{[2]}}\}$. Let $\mathbf{M}$ be the submodule of $\mathbf{F}$ generated by

$$\{x_1 x_2 e_\pi, (x_1 + x_2)e_\sigma + x_3 e_\tau\} \subset \mathbf{F}_3$$

(see chalkboard for maps). Then with Macaulay2 we compute the beginning of a free resolution (the minimal resolution) of $\mathbf{M}$:

$$\mathbf{G}^4 \to \mathbf{G}^3 \to \mathbf{G}^2 \to \mathbf{G}^1 \to \mathbf{G}^0 \to \mathbf{M} \to 0$$

where

$$\mathrm{rk}(\mathbf{G}^4) = 20,\ \mathrm{rk}(\mathbf{G}^3) = 13,\ \mathrm{rk}(\mathbf{G}^2) = 8,\ \mathrm{rk}(\mathbf{G}^1) = 4,\ \mathrm{rk}(\mathbf{G}^0) = 2.$$

`https://github.com/morrowmh/OIGroebnerBases`

References

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